Solution to the N-Player Gambler's Ruin with Variable Bet Sizes Using Recursions Based on Multigraphs

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ABSTRACT

This paper studies the N -player gambler's ruin with variable bet size, that is, more than one chip may be transferred from one player to another. Weighted directed multi- graphs were constructed to describe the transitions between chip states. Linear systems were constructed based on the connections between nodes in these graphs. Solutions for the placing probabilities of each player are obtained from these linear systems. Expected time until ruin is solved by modeling the game as a Markov process. A numerical algorithm was developed to solve the N -player gambler's ruin with variable bet size for any positive integer chip total. Compared with the classic N -player gambler's ruin, game durations are expected to be shorter due to the nature of bet sizes allowed in this model. Expected time until ruin and placing probabilities for All-In betting games are shown to be dependent only on the wealth proportion of the players.

INTRODUCTION

Consider a game with N players having initial wealths S_1, S_2,..., S_N. The N -player gambler's ruin problem is finding each player's winning probability and the game's expected duration with the assumption that they are playing a fair game. The N -player game has several variations based on the players involved in each betting round, and how winners and losers are selected in these rounds. One of the common variants is the N -tower game, where each round has one winner and one loser. Each game involves exactly two of the N players, say Players i and j, each with an equal probability of winning the game. The players are paired randomly with equal probabilities, so that each pair has a probability $\frac{2}{N(N-1)}$ of being selected. In the classic (or Unit Bet) variant, bet size is fixed to 1 unit for each game. In this paper, we will focus on variants where players can wager up to their whole wealth, depending on the wealth of their opponent. These variants are the All-In and No Limit (or "occasionally all in") variations, as described in a paper by Diaconis and Ethier [6].

In the past years, several models have been used in solving the N-player gambler's ruin where $N > 2$. In three-player problems, first studied by Bachelier in 1912 [2], the player's winning probability is solved by recursion and is given by the proportion of that player's wealth to the total wealth of the three players involved in the game. E Kim (2005) discussed further results for the three-player problem, including poker applications [3].

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Playing strategies were studied to determine optimal results depending on the payout structure of the tournament. David (2015) used multigraphs and recursions in solving the three-player problem [4]. $VVVXT = \frac{3S_1S_2S_3}{S_1+S_2+S_3}$ [2, 7, 16, 17]. Stirzaker introduced solutions using martingales in this area of research [16]. Swan and Bruss (2006) described a matrix-analytic approach in solving the N-player problem [18], wherein they solved for ruin probabilities for each player. Diaconis et al. (2020) estimated gambler's ruin on finite inner uniform domains, with the three-player problem as an example [5]. Marfil and David (2020) used weighted directed multigraphs and recursions in solving the four-player problem [13] and eventually generalized the solution for the N-player problem [14]. Diaconis and Ethier (2022) discussed several methods of approximating gambler's ruin probabilities [6]. While exact computations are feasible for small wealth totals, these Markov chain methods seem difficult for values of practical interest. Ganzfried and Sandholm (2008) studied N-player gambler's ruin with all-in betting with applications in the poker context [9].

The main purpose of this paper is to solve the placing probabilities of each player for the N-player gambler's ruin problem with varying bet sizes (No Limit and All-In variants) and $N > 3$. Here, placing probabilities refer to a player's probability of finishing in first, second, up to the last place. Specifically, the paper aims to accomplish the following. First, we will construct a weighted directed multigraph for the N-player game variants with prescribed total wealth S. From this, we will construct an appropriate linear system that represents the multigraph for each game variants. We will then use a numerical algorithm for solving placing probabilities for each game variant. We will present a method for calculating the expected time until ruin for each game variant. Finally, we will compare our results with existing results for the classic N-player game variant.

MATERIALS AND METHODS

Weighted directed multigraphs will be used to model transitions between states given the players' initial wealths. Variables for the players' placing probabilities will be assigned to each unique state. The system of equations relating these variables will then be solved. The problem will be modeled using an absorbing Markov chain in order to calculate the expected time until ruin, as shown in Section 2.4. For definitions of weighted directed multigraphs and absorbing Markov chains, please see previous papers by Marfil and David, and Grinstead's Introduction to Probability [13, 15, 10].

Definition of States

This paper aims to calculate placing probabilities for the N players given their initial wealths. Probabilities of placing first are just the proportions of the players' initial wealths to the total wealth in play [2]. We are left to calculate the remaining placing probabilities for each player, which we accomplish using a numerical algorithm (in Matlab) described in this section. We first present the following.

Definition 1 *A **chip state** is an ordered N-tuple* (S_1, S_2, ..., S_N), *where* $S_1, S_2, \ldots, S_N \in \mathbb{N} \cup \{0\}$ *and* $S_1 \geq S_2 \geq \ldots \geq S_N$. *The coordinates of a chip state represent the wealths of each player at a given time.*

Definition 2 *A **chip position** is a coordinate of a chip state* (S_1, S_2, ..., S_N). *Each position in a chip state has corresponding placing probabilities.*

Definition 3 *A **terminal state** is a chip state where at least one chip position is zero. Placing probabilities for each player for terminal states can be solved based on the solutions to the* $(N-1)$*-player gambler's ruin problem.*

Definition 4 *A **nonterminal state** is a chip state with N positive chip positions.*

Definition 5 *In a betting round involving players with wealths x and y, for the No Limit game variant, the **permissible bet sizes** are elements of the set* $\{1, 2, \ldots, \min\{x, y\}\}$, *or natural numbers less than or equal to the number of chips in the smaller stack involved in the betting round. For the All-In game variant, the bet size is* $\min\{x, y\}$.

Two players are selected randomly using a uniform distribution, and these players face each other in an even-money betting round. Without loss of generality, let the two players be Players 1 and 2 having wealths x and y, respectively. For the All-In variant, the bet size n is $\min\{x, y\}$. For the No Limit variant, the bet size n is selected randomly using a uniform distribution from the set of permissible bet sizes $\{1, 2, \ldots, \min\{x, y\}\}$. The winner of the round is selected randomly, and that player adds to his stack n chips taken from the other player's initial stack. For example, if Player 1 wins over Player 2, their new chip stacks will be $x + n$ and $y - n$, respectively. Upon reaching a terminal state, a chip state of the form $(a, b, 0)$, second place probabilities can be solved trivially. Upon reaching a nonterminal state, the process is repeated.

We use the following definition for the mapping for the N-player gambler's ruin.

Definition 6 *Consider a nonterminal state* (S_1, S_2, ..., S_N) *for the N-player gambler's ruin for a given wealth total S. The possible states after one round of betting are of the form:*

$$(S_1, S_2, \ldots, S_i+n_{ij}, \ldots, S_j-n_{ij}, \ldots, S_N), \quad (1)$$

where $i, j, \in \{1, 2, \ldots, N\}$, $i \neq j$ *and* n_{ij} *is the bet size. For the All-In variant, each state has probability* $\frac{1}{N(N-1)}$ *of being selected. For the No Limit variant, each state has probability* $\frac{1}{N(N-1)m}$ *of being selected, where m is the maximum permissible bet size between the players participating in the round.*

Construction of Multigraph

Let S be the total wealth of the N players. We construct the weighted directed multi- graph using the following algorithm:

1. Unique states (S_1, S_2, ..., S_N) are generated such that $S = S_1 + S_2 + \cdots + S_N$ and $S_1 \geq S_2 \geq \ldots \geq S_N$. The unique states will serve as the nodes in the graph. By convention, states are arranged in columns according to their last coordinates.

2. Terminal states (with exactly one zero coordinate) are put on the leftmost column, with the states arranged in decreasing order of the first coordinates.

3. States with the last coordinate equal to 1 are then put on the next column, with thestates also arranged in decreasing order of the first coordinates.

4. The process of constructing this lattice is repeated until all states are exhausted.

5. Given a nonterminal state (S_1, S_2, ..., S_N), if it is possible for the chip stacks to become (R_1, R_2, ..., R_N)

after one round of betting, an edge directed from (S_1, $S_2, \ldots, S_N$) to ($R_1, R_2, \ldots, R_N$) is constructed, with weight as defined in 6, so that the total weight of all outward edges from a nonterminal state must be 1.

6. Loops may be constructed if a state goes to itself (up to permutation of coordinates) after one round. Multiple edges between a pair of nodes may also be constructed if there are multiple possible ways of transitioning between these state pairs.

Construction of Linear System

For the N-player gambler's ruin for a given wealth total S, we construct the linear system using the following algorithm: [15]

1. *Variable assignment*
   A variable will be assigned to each of the unique chip positions from all terminal andnonterminal states in the order they are generated in 2.2.

2. *Construction of transition matrix* $\mathbf{Q}$
   If n is the number of unique chip positions from all nonterminal states, an $n \times n$ matrix $\mathbf{Q}$ representing the transitions among nonterminal chip positions is constructed.

3. *Construction of transition matrix* $\mathbf{R}$
   If m is the number of unique chip positions from all terminal states, an $n \times m$ matrix $\mathbf{R}$ representing the transitions from nonterminal to terminal chip positions is also constructed.

4. *Computation of entries of matrices* $\mathbf{Q}$ *and* $\mathbf{R}$
   For each of the n variables corresponding to nonterminal positions, we determine wherethe corresponding chip positions are being moved.

5. *Set up of linear system*
   We set up and solve the linear system

$$(\mathbf{I} - \mathbf{Q})\mathbf{B} = \mathbf{R}, \tag{2}$$

   where $\mathbf{B}$ is the absorption probability matrix, an $n \times m$ matrix containing the probabilities for the n unique nonterminal chip positions of ending up in the m unique terminal chip positions.

6. *Recursion for* $(N-1)$*-player gambler's ruin*
   After solving for matrix $\mathbf{B}$, we construct the terminal placing probability matrix $\mathbf{W}$, an $m \times N$ matrix containing the placing probabilities for the corresponding terminal positions in the N-player game. We do this by recursively applying the above algorithm to the $(N-1)$-player gambler's ruin until we reach the two-player case. Specifically, matrix $\mathbf{W}$ is obtained by adding rows for zero positions and a column for Nth place probabilities to the matrix $\mathbf{B_{N-1}}$, or the absorption probability matrix for the $(N-1)$-player case.

7. *Computation of placing probabilities*
   Finally, we recursively solve up to the given N to find the placing probability matrix $\mathbf{BW}$, which contains the final placing probabilities.

Given total wealth S, one can solve for m and n using integer partitions [1] and recurrence relations. These equations are shown in a paper by Marfil and David but no closed formulas were provided for m and n [14].

Calculation of Expected Time until Ruin

In the N-player gambler's ruin, time until ruin is the number of games played by the players until one player reaches zero wealth. Recall that an absorbing Markov chain $\mathbf{P}$ has the following canonical form:

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{R} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix} \tag{3}$$

where $\mathbf{Q}$ represents the transitions among nonterminal states, $\mathbf{R}$ represents the transitions from nonterminal states to terminal states, and $\mathbf{I}$ represents the transitions among terminal states. For the canonical form of the matrix, we use matrices $\mathbf{Q}$ and $\mathbf{R}$ as constructed in Section 2.3, while $\mathbf{I}$ is an m-by-m identity matrix.

In modeling this problem as an absorbing Markov chain, the chain is absorbed when one of the N players is ruined. Hence, the expected time until ruin is equal to the time until absorption of the chain. We use the fundamental matrix $\mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}$ of the absorbing Markov chain $\mathbf{P}$, with the existence of the former guaranteed [10, 15]. Consequently, this also guarantees the existence of solutions for our linear system for ruin probabilities. The expected time until ruin is evaluated using the following theorem [10].

Theorem 1 *Given an absorbing Markov chain* $\mathbf{P}$ *with* $\mathbf{N}$ *as its fundamental matrix, and* $\mathbf{P}$ *starts in state* s_i*, let* T_i *be the expected number of steps before the chain is absorbed. Let* $\mathbf{T}$ *be the column vector whose* i^{th} *entry is* T_i*. Then*

$$\mathbf{T} = \mathbf{N} \cdot \mathbf{1}, \tag{4}$$

where $\mathbf{1}$ *is the column vector whose entries are all 1.*

The following theorem tells us how absorption probabilities are obtained using the fun- damental matrix of an absorbing Markov chain [10, 13, 15].

Theorem 2 *Let* b_{ij} *be the probability that an absorbing chain will be absorbed in the ab- sorbing stateа* s_j *if it starts in the transient state* s_i*. Let* $\mathbf{B}$ *be the matrix with entries* b_{ij}*. Then* $\mathbf{B}$ *is a* $t \times r$ *matrix, and*

$$\mathbf{B} = \mathbf{NR}, \tag{5}$$

where $\mathbf{N}$ *is the fundamental matrix and* $\mathbf{R}$ *is as in the canonical form.*

RESULTS

In general, for any positive integer S, the system can be modeled as a multigraph with loops where vertices are unique states up to permutations and directed edges represent the transitions between states.

Multigraph Theorems

The following theorems describe the connections between a nonterminal state (S_1, S_2, ..., S_N) and some nonterminal and terminal states.

Theorem 3 *For the N-player No Limit gambler's ruin for a given wealth total S, given a nonterminal state* (S_1, S_2, ..., S_N)*, where* $S = S_1 + S_2 + \cdots + S_N$ and $S_1 \geq S_2 \geq \ \ldots \geq S_N \geq 1$*, it is connected with edge weights* $\frac{1}{N(N-1)m}$ *to states of the form*

$$(S_1, S_2, \ldots, S_i + n_{ij}, \ldots, S_j - n_{ij}, \ldots, S_N), \tag{6}$$

where $i, j, \in \{1, 2, \ldots, N\}$, $i \neq j$ *and* $m = \min\{S_i, S_j\}$ *is the maximum permissible bet size, and* $n_{ij} \in \{1, 2, \ldots, m\}$ *is the bet size. The chip positions are rearranged in decreasing order to get a valid chip state.*

Proof: Let $(S_1, S_2, \ldots, S_N)$ be a nonterminal state. Since players are selected randomly using a uniform distribution and there is a winner and a loser in each round, there are N ways of selecting a winner and $N-1$ ways of selecting a loser from the remaining players. Suppose Player i is selected as the winner, while Player j is selected as the loser. In each round, the maximum permissible bet size is $m = \min\{S_i, S_j\}$, with the winner gaining n_{ij} chips from the loser. Then from Definition 6, the transition after one round is

$$(S_1, S_2, \ldots, S_i, \ldots, S_j, \ldots, S_N) \rightarrow (S_1, S_2, \ldots, S_i + n_{ij}, \ldots, S_j - n_{ij}, \ldots, S_N).$$

Since there are $N(N-1)$ ways of selecting these pairs of players, each with equal probability, and there are m possible bet sizes, each outward edge from this nonterminal state has weight $\frac{1}{N(N-1)}$.

Theorem 4 *For the N-player All-In gambler's ruin for a given wealth total S, given a nonterminal state* $(S_1, S_2, \ldots, S_N)$, *where* $S = S_1 + S_2 + \cdots + S_N$ and $S_1 \geq S_2 \geq \ldots \geq S_N \geq 1$, *it is connected with edge weights* $\frac{1}{N(N-1)}$ *to states of the form*

$$(S_1, S_2, \ldots, S_i + n_{ij}, \ldots, S_j - n_{ij}, \ldots, S_N), \tag{7}$$

where $i, j, \in \{1, 2, \ldots, N\}$, $i \neq j$ *and* $n_{ij} = \min\{S_i, S_j\}$ *is the bet size. The chip positions are rearranged in decreasing order to get a valid chip state.*

Proof: The proof follows similarly with fixed bet size $n_{ij} = \min\{S_i, S_j\}$ for a pair of players S_i and S_j. Since there's only one possible bet size for each pair of players and there are $N(N-1)$ equally probable ways of selecting these pairs of players, each outward edge from this nonterminal state has weight $\frac{1}{N(N-1)}$.

Four-Player Game for S=6

Example 1 *In this example, we take a more detailed look at the four-player results for S = 6. There are two ways of distributing six units of wealth to four players such that all of them have positive integer wealth totals:* (3, 1, 1, 1) *and* (2, 2, 1, 1), *up to permutation of elements. There are also three ways of distributing six units of wealth to four players such that all of them have nonnegative integer wealth totals, with one player having zero wealth:*(4, 1, 1, 0), (3, 2, 1, 0) *and* (2, 2, 2, 0), *up to permutation of elements. We look at the graph for the four-player No Limit game with total wealth S = 6, as shown in Fig. 1.*

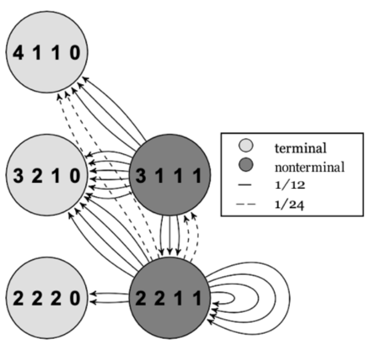

Figure 1: Multigraph for $N = 4$, $S = 6$ **No Limit game**

In this example, the unique terminal states are (4, 1, 1, 0), (3, 2, 1, 0) *and* (2, 2, 2, 0), *de- noted by* P_1, P_2, *and* P_3, *respectively. The unique nonterminal states are* (3, 1, 1, 1) *and* (2, 2, 1, 1), *denoted by* P_4, *and* P_5, *respectively. From these states, we observe that there are four unique chip positions in nonterminal states and 9 unique chip positions in terminal states. Also, notice that edges have weights 1/6 or 1/12. This difference in edge weights occurs due to the different permissible bet sizes in each scenario. Four loops are present for the state* P_5 *since it can transition to itself if Player 3 or 4 beats Player 1 or 2 for one chip. After the construction of a multigraph for a given wealth total S, a linear system rep- resenting the transitions between the states is constructed, as detailed in Section 2.3. For example, if* $S = 6$, *we define four variables* v_1, v_2, v_3, *and* v_4 *corresponding respectively to the unique nonterminal chip positions 3 and 1 in* P_4, *and positions 2 and 1 in* P_5. *We also define nine variables* w_1, w_2, *up to* w_9 *for terminal states, corresponding respectively to the chip positions 4, 1, and 0 in* P_1, *positions 3, 2, 1, and 0 in* P_2, *and positions 2 and 0 in* P_3.

We get the linear system by setting up the recurrence:

$$\left(\mathbf{I}_4 - \frac{1}{12}\begin{bmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 \\ 0.5 & 0.5 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix}\right)\mathbf{B} = \frac{1}{12}\begin{bmatrix} 3 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 & 2 & 2 & 0 & 0 \\ 0.5 & 0 & 0.5 & 2 & 2 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 2 & 1 & 1 \end{bmatrix} \tag{8}$$

The solution to this system is an $n \times m$ *matrix that gives the absorption probabilities for the n unique nonterminal chip positions into the m unique terminal chip positions.*

This system has the following solution:

$$\mathbf{B} = \frac{1}{12}\begin{bmatrix} 0.26660 & 0.0088 & 0.0149 & 0.5599 & 0.0564 & 0.0144 & 0.0144 & 0.0589 & 0.0056 \\ 0.0079 & 0.1906 & 0.0918 & 0.0284 & 0.1963 & 0.2102 & 0.2102 & 0.0449 & 0.0196 \\ 0.0665 & 0.0352 & 0.0596 & 0.2396 & 0.2256 & 0.0.057 & 0.0.057 & 0.2355 & 0.0226 \\ 0.0141 & 0.1261 & 0.0211 & 0.0508 & 0.0647 & 0.2326 & 0.2326 & 0.1516 & 0.1065 \end{bmatrix} \tag{9}$$

which means that for v_1 *(3 in* P_4*), its probability of ending up in the terminal chip position* w_1 *(4 in* P_1*) is* 26.66%, *and so on. Note that* w_3, w_7, *and* w_9 *represent ruined chip positions for the four-player game. This implies that* v_1*'s ruin probability is given by the sum*

$$B_{1,3} + B_{1,7} + B_{1,9} = 3.49\%. \tag{10}$$

To calculate the players' placing probabilities, we need the solutions to the three-player problem for S = 6. The graph for the three-player game with total wealth S = 6 is shown in Fig. 2.

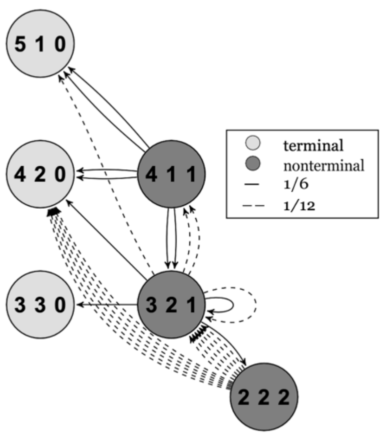

Figure 2: Multigraph for $N = 3$, $S = 6$ **No Limit game**

The algorithm applied previously is then applied recursively to this three-player game. As shown by Marfil and David (2021) [15], the absorption probabilities for three players are given by the matrix

$$\mathbf{C} = \begin{bmatrix} 0.3843 & 0.0112 & 0.0136 & 0.4362 & 0.0343 & 0.0295 & 0.0847 & 0.0062 \\ 0.0124 & 0.1989 & 0.1977 & 0.0319 & 0.2329 & 0.2353 & 0.0486 & 0.0423 \\ 0.1528 & 0.0336 & 0.0409 & 0.3087 & 0.1029 & 0.0884 & 0.2540 & 0.0187 \\ 0.0553 & 0.0505 & 0.1215 & 0.1185 & 0.2617 & 0.1197 & 0.2251 & 0.0476 \\ 0.0192 & 0.1431 & 0.0649 & 0.0728 & 0.1354 & 0.2918 & 0.0663 & 0.2064 \\ 0.0379 & 0.0379 & 0.0379 & 0.2500 & 0.2500 & 0.2500 & 0.0909 & 0.0455 \end{bmatrix} \tag{11}$$

while a corresponding matrix

$$\mathbf{X} = \begin{array}{c} \\ 5 \\ 1 \\ 0 \\ 4 \\ 2 \\ 0 \\ 3 \\ 0 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 5/6 & 1/6 & 0 \\ 1/6 & 5/6 & 0 \\ 0 & 0 & 1 \\ 2/3 & 1/3 & 0 \\ 1/3 & 2/3 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \tag{12}$$

has entries which are the placing probabilities for each nonterminal chip position in the three-player game. Note that the elements of matrix $\mathbf{X}$ *are obtained from the two-player base case wherein a player's win probability is just the proportion of that player's wealth to the total wealth of both players.*

The placing probabilities for the three-player game are the elements of the matrix shown in Eq. (13):

$$\mathbf{CX} = \begin{array}{c} \\ 4 \\ 1 \\ 3 \\ 2 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.6667 & 0.2840 & 0.0493 \\ 0.1667 & 0.3580 & 0.4753 \\ 0.5000 & 0.3520 & 0.1480 \\ 0.3333 & 0.3779 & 0.2888 \\ 0.1667 & 0.2702 & 0.5632 \\ 0.3333 & 0.3333 & 0.3333 \end{bmatrix} \end{array} \tag{13}$$

From this matrix, we get matrix $\mathbf{W}$ *for the four-player game by adding rows for zero positions and a column for fourth place probabilities, as shown in Eq. (14):*

$$\mathbf{W} = \begin{array}{c} \\ 4 \\ 1 \\ 0 \\ 3 \\ 2 \\ 1 \\ 0 \\ 2 \\ 0 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.6667 & 0.2840 & 0.0493 & 0 \\ 0.1667 & 0.3580 & 0.4753 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5000 & 0.3520 & 0.1480 & 0 \\ 0.3333 & 0.3779 & 0.2888 & 0 \\ 0.1667 & 0.2702 & 0.5632 & 0 \\ 0 & 0 & 0 & 1 \\ 0.3333 & 0.3333 & 0.3333 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{14}$$

Finally, using $\mathbf{B}$ *in Eq. (9) and* $\mathbf{W}$ *in Eq. (14), we get matrix* $\mathbf{BW}$*, whose entries are the placing probabilities for each nonterminal chip position in the four-player game. This matrix is shown in Eq. (15):*

$$\mathbf{BW} = \begin{array}{c} \\ 3 \\ 1 \\ 2 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.5000 & 0.3208 & 0.1443 & 0.0350 \\ 0.1667 & 0.2264 & 0.2852 & 0.3217 \\ 0.3333 & 0.2952 & 0.2316 & 0.1399 \\ 0.1667 & 0.2048 & 0.2684 & 0.3601 \end{bmatrix} \end{array} \tag{15}$$

In calculating the expected time until ruin, we use the fundamental matrix for $N = 4$ *and* $S = 6$*:*

$$\mathbf{N} = \left(\mathbf{I_4} - \frac{1}{12} \begin{bmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 \\ 0.5 & 0.5 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix} \right)^{-1} \tag{16}$$

Hence,

$$\mathbf{T} = \mathbf{N} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 44/31 \\ 44/31 \\ 52/31 \\ 52/31 \end{bmatrix} \tag{17}$$

which gives the expected time until ruin for the nonterminal states for $S = 6$. *This means that the expected time until ruin for the states* (3, 1, 1, 1) *and* (2, 2, 1, 1) *are* 44/31 *and* 52/31 *time steps, respectively.*

Example 2 *For comparison, the graph for the four-player All-In game with total wealth* $S = 6$ *is shown in Fig. 3.*

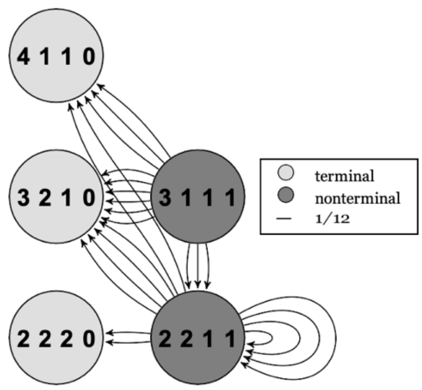

Figure 3: Multigraph for $N = 4$, $S = 6$ All-In game

Notice that all edges have the same weight 1/12 *since each possible player pair yields only two possible outcomes each. This means that multigraph construction is easier for All-In cases than for No Limit cases, where edge weights vary due to different possible bet sizes.*

Applying the same algorithm in Example 1, matrix **BW** *in Eq. (18) below contains the placing probabilities for each nonterminal chip position in the four-player All-In game:*

$$BW = \begin{array}{c} \\ 3 \\ 1 \\ 2 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.5000 & 0.3382 & 0.1336 & 0.0282 \\ 0.1667 & 0.2206 & 0.2888 & 0.3239 \\ 0.3333 & 0.3156 & 0.2384 & 0.1127 \\ 0.1667 & 0.1844 & 0.2616 & 0.3873 \end{bmatrix} \end{array} \quad (18)$$

Computing the expected time until ruin for the states (3, 1, 1, 1) *and* (2, 2, 1, 1) *yields 22/15 and 28/15 time steps, respectively. These are shorter expected times compared with the No Limit game since there is a higher probability of player ruin due to the nature of betting.*

Expected Time until Ruin

Table 1 shows the expected time until ruin for $S = 5, 6, 7$ and 8 for the Unit Bet, No Limit, and All-In four-player gambler's ruin scenarios.

Table 1: Expected Time Until Ruin for States with $N = 4$ and $S = 5$, 6, 7 and 8

		Expected Time Until Ruin		
S	State	Unit Bet	No Limit	All-In
5	(2,1,1,1)	1.333333	1.333333	1.333333
6	(3,1,1,1)	1.466667	1.419355	1.375000
	(2,2,1,1)	1.866667	1.677419	1.500000
7	(4,1,1,1)	1.528090	1.456311	1.401709
	(3,2,1,1)	2.112360	1.825243	1.606838
	(2,2,2,1)	2.741573	1.941748	1.333333
8	(5,1,1,1)	1.559578	1.476520	1.424419
	(4,2,1,1)	2.238311	1.906079	1.697674
	(3,3,1,1)	2.437406	1.922619	1.546512
	(3,2,2,1)	3.193062	2.204092	1.639535
	(2,2,2,2)	4.193062	2.102046	1.000000

Expected time until ruin for all three variations are calculated by getting the sum of row entries in the corresponding fundamental matrix **N** for each wealth total. An example of this is shown in Eq. 17 for states with $S = 6$.

Observe that for any state, the expected time until ruin for the Unit Bet variation is always the largest, while the All-In variation has the smallest expected time until ruin. This is true because of the difference in nature of the transfer of wealth between pairs of players in each round of the three cases. For the Unit Bet variation, exactly one chip is transferred between the players in each round. This means that if all players have more than one chip, it is impossible for a player to run out of chips in a single round. Only players with one chip at the start of the round are at risk of being ruined immediately. States of this form are sometimes called near-terminal states since they can go to a terminal state in a single step. For the No Limit variation, more than one chip may be transferred between the players, with the bet size chosen uniformly. In fact, all the chips of the player with the smaller wealth may be transferred to the other player in a single round. This results in a higher probability of player ruin in a single round. Hence, a player may achieve ruin faster, leading to a lower expected time until ruin. Also, if the two players involved in the round have equal wealth totals, each of them has a nonzero probability of achieving ruin in a single round. Lastly, for the All-In variation, the smaller stack in each round is always at risk of being ruined immediately. Moreover, if the two players involved in the round have the same stack, one of them will certainly be ruined after that round. This leads to the shortest expected time until ruin among the three variations. In fact, if all players have the same wealth total at the start of the round, any combination of players will lead to a player being ruined after that round, leading to the shortest expected ruin time of 1 time step. This is in contrast to the Unit Bet scenario where the case of all players having equal wealth always leads to the largest expected ruin time among all possible cases for that wealth total.

Table 2 shows the expected time until ruin for $S = 12$ for the Unit Bet, No Limit, and All-In four-player gambler's ruin scenarios.

Table 2: Expected Time Until Ruin for States with $N = 4$ and $S = 12$

	Expected Time Until Ruin		
State	Unit Bet	No Limit	All-In
(9,1,1,1)	1.598230	1.494194	1.429599
(8,2,1,1)	2.392919	1.976777	1.718396
(7,3,1,1)	2.832561	2.130154	1.741366
(6,4,1,1)	3.062454	2.197816	1.763273
(5,5,1,1)	3.134625	2.175915	1.621409
(7,2,2,1)	3.749197	2.428346	1.710345
(6,3,2,1)	4.518483	2.678824	1.856213
(5,4,2,1)	4.872647	2.747851	1.864226
(5,3,3,1)	5.433894	2.793086	1.740148
(4,4,3,1)	5.725927	2.827240	1.708725
(6,2,2,2)	6.004048	2.749855	1.375000
(5,3,2,2)	7.230028	3.111520	1.715134
(4,4,2,2)	7.621062	3.101327	1.500000
(4,3,3,2)	8.512561	3.272736	1.739080
(3,3,3,3)	9.512561	3.021941	1.000000

Note that the states are arranged in increasing order according to the fourth coordinates, then the third coordinates, and then the second coordinates. Observe that the expected time until ruin for the Unit Bet case is increasing as we go down the column. Recall that the expected time until ruin for a state (*A*, *B*, *C*) in the three-player Unit Bet case is given by the formula $T = \frac{3ABC}{A+B+C}$ [2, 7, 16]. In this formula, expected time until ruin is maximized when all players start with equal wealths. Although no formula and proof has been obtained for expected time until ruin for the Unit Bet case for $N > 3$ players, we can observe that the expected ruin time similarly increases as the wealth totals among players become more balanced.

Now, observe that while the expected time until ruin for the No Limit case is generally increasing as we go down the column, there are some cases when the expected time decreases. For $S = 12$, let's look at two specific states: (5, 5, 1, 1) and (6, 4, 1, 1).

For the state (5, 5, 1, 1), notice that the top two players have the same wealth total. We compare this with the previous state (6, 4, 1, 1), which has three non-equal positions. For the state (5, 5, 1, 1), all players can be ruined in a single round. For example, the first and second players can be ruined if they face each other in the round, with bet size equal to 5. Players 3 and 4 can be ruined in any round. Calculating the probability of a player being ruined in one round, we have $2\left(\frac{1}{12}\right)\left(\frac{1}{5}\right) + 6\left(\frac{1}{12}\right)(1) = \frac{8}{15}$. For the state (6, 4, 1, 1), Player 1 has zero probability of being ruined in a single round since the maximum permissible bet sizes involving them is either 1 or 4. Only Players 2, 3 or 4 can be ruined in a single round. Calculating the probability of a player being ruined in one round, we have $1\left(\frac{1}{12}\right)\left(\frac{1}{4}\right) + 6\left(\frac{1}{12}\right)(1) = \frac{25}{48}$, which is smaller than $\frac{8}{15}$. Hence, it is more likely that a player is ruined in the first round for the state (5, 5, 1, 1), leading to a smaller expected time until ruin for this state.

For the All-In case, observe that (3, 3, 3, 3) has the smallest expected ruin time, while the (6, 3, 2, 1) and (5, 4, 2, 1) largest expected ruin times. As discussed earlier, if all players have the same wealth total at the start of the round, any combination of players will lead to a player being ruined after that round, leading to the shortest possible ruin time. On the other hand, if all players have different wealth totals, only the smaller stack in each round is at risk of ruin in that round. Looking at all possible cases for $S = 12$, only (6, 3, 2, 1) and (5, 4, 2, 1) have four different wealth totals for the players. Those with three different wealth totals (among four players) are in the next tier in terms of expected ruin time.

Table 3 shows the expected time until ruin for some states of the form (4*k*, 3*k*, 2*k*, *k*) for the Unit Bet, No Limit, and All-In four-player gambler's ruin scenarios.

Table 3: Expected Time Until Ruin for Some States of the Form (4*k*, 3*k*, 2*k*, *k*)

	Expected Time Until Ruin		
State	Unit Bet	No Limit	All-In
(4,3,2,1)	4.150468	2.551239	1.825054
(8,6,4,2)	16.720608	4.338951	1.825054
(12,9,6,3)	37.669695	5.622064	1.825054
(16,12,8,4)	66.998232	6.613142	1.825054
(20,15,10,5)	104.706296	7.430748	1.825054

For the Unit Bet case, as the wealth total increases while keeping the proportions con- stant, the expected ruin time increases almost proportionally. For the No Limit case, while expected ruin time increases with an increase in total wealth among players, the increase in ruin time is very small compared to the previous case. For the All-In case, if the wealth total increases while proportions among players are kept constant, the expected ruin time does not change at all. These differences in expected ruin time reflect the different bet sizes among the models. For the Unit Bet case, the bet size is fixed at 1 unit. Increasing the wealth totals but keeping the bet size at 1 unit clearly yields longer game durations. For the No Limit case, although the ruin time increases with the wealth total, the rate of increase is much slower since at least one player (the shorter stack) can be ruined in each round if the bet size is equal to their stack size. Lastly, for the All-In case, since the bet size is equal to the shorter stack between the selected players, increasing the wealth total yields larger bet sizes. This results in ruin time depending only on the wealth proportions and not on the total among players.

Theorem 5 explains the existence of a lower bound and an upper bound for the expected ruin time for the All-In scenario.

Theorem 5 *For the N-player All-In gambler's ruin for a given wealth total S, for any initial state, the expected time until ruin lies on the interval* [1, 2].

Proof: Suppose N players have total wealth S, with each player having positive initial wealth. Clearly, the game takes at least one round since each player has positive wealth. Ruin is certainly achieved when each player has equal wealth, such that any pairing will result in one player losing their wealth in that round. Hence, the greatest lower bound for expected ruin time is 1. On the other hand, the longest games are feasible when each possible pair has players with unequal wealth, with the shorter stack winning each round. In this case, the shorter stack has 0.5 probability of being ruined in each round. For this scenario, if T is the ruin time, we have

$$E[T] = 0.5(1) + 0.5^2(2) + 0.5^3(3) + \cdots = \sum_{n=1}^{\infty} n(0.5)^n = 2.$$

Hence, the expected time until ruin lies on the interval [1, 2].

Comparison with ICM

We compare the placing probabilities for some states of the form (4*k*, 3*k*, 2*k*, *k*) using the No-Limit model, All-In model, and the Independent Chip Model (ICM) [12], which is a widely used but simple approximation used in poker. Table 4 shows these placing probabilities. First place probabilities are the same for all models. Observe that the solutions obtained from the No Limit model depend on the actual number of chips of each player while the All-In model and ICM depend only on the proportion of the chip stacks, which means that all states of the form (4*k*, 3*k*, 2*k*, *k*) where $k \in \mathbb{N}$ have the same placing probabilities using either the All-In model or ICM. However, placing probabilities obtained using the All-In model and ICM are not equal.

Table 4: Solutions for Some States of the Form $(4k, 3k, 2k, k)$

State	Position	$P(X_i = 1)$	$P(X_i = 2)$	$P(X_i = 3)$	$P(X_i = 4)$
(4,3,2,1)	4	0.4000	0.3125	0.2055	0.0820
(8,6,4,2)	8	0.4000	0.3052	0.2051	0.0896
(12,9,6,3)	12	0.4000	0.3010	0.2047	0.0943
(16,12,8,4)	16	0.4000	0.2982	0.2044	0.0974
(20,15,10,5)	20	0.4000	0.2960	0.2042	0.0998
All-In	4k	0.4000	0.2926	0.2101	0.0972
ICM	4k	0.4000	0.3159	0.2063	0.0778
(4,3,2,1)	3	0.3000	0.2937	0.2534	0.1530
(8,6,4,2)	6	0.3000	0.2899	0.2506	0.1596
(12,9,6,3)	9	0.3000	0.2875	0.2491	0.1634
(16,12,8,4)	12	0.3000	0.2859	0.2481	0.1661
(20,15,10,5)	15	0.3000	0.2847	0.2473	0.1680
All-In	3k	0.3000	0.2702	0.2414	0.1884
ICM	3k	0.3000	0.3083	0.2619	0.1298
(4,3,2,1)	2	0.2000	0.2449	0.2918	0.2632
(8,6,4,2)	4	0.2000	0.2468	0.2878	0.2654
(12,9,6,3)	6	0.2000	0.2478	0.2857	0.2665
(16,12,8,4)	8	0.2000	0.2485	0.2842	0.2673
(20,15,10,5)	10	0.2000	0.2490	0.2832	0.2678
All-In	2k	0.2000	0.2433	0.2667	0.2900
ICM	2k	0.2000	0.2413	0.3175	0.2413
(4,3,2,1)	1	0.1000	0.1674	0.2633	0.4693
(8,6,4,2)	2	0.1000	0.1581	0.2565	0.4854
(12,9,6,3)	3	0.1000	0.1636	0.2606	0.4758
(16,12,8,4)	4	0.1000	0.1674	0.2633	0.4693
(20,15,10,5)	5	0.1000	0.1703	0.2653	0.4644
All-In	k	0.1000	0.1939	0.2818	0.4243
ICM	k	0.1000	0.1345	0.2143	0.5512

CONCLUSION

In this paper, a method of solving placing probabilities for two variations of the *N*-player gambler's ruin was presented. The assumptions for the problem were that betting was even-money with no draw, bet size is either the shorter stack or is uniformly chosen among all permissible integer bet sizes, and the participating players for each round were selected randomly following a uniform distribution. The method used recursions with players having integer wealths at the start of each round. A multigraph for the different states for a fixed total wealth S was constructed and a linear system was constructed to represent the transition between these states. Solutions of this linear system give the absorption probabilities while placing probabilities can be obtained by applying the algorithm recursively. Expected time until ruin for any given state can be solved using this model, but a formula has yet to be obtained. Using the presented algorithm gives exact solutions to the problem, as opposed to numerical methods yielding approximations. Among the N-player variations studied, only the All-In variation yields solutions depending only on the wealth proportions and not on the total wealth among all players.

The model may be extended to one wherein the bet size selection may follow distributions other than the uniform distribution. This may lead to a better approximation of most poker tournaments since bet size between players usually belongs to the lower range of permissible bet sizes. The model may also be applied to other variations of the N-player game such at the player-centric and symmetric games. A formula for the expected time until ruin for $N > 3$ for the Unit Bet, No Limit and All-In variants may be obtained.

DECLARATIONS

Competing Interests
The author declares no competing interests.

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REFERENCES

[1] Andrews, G.E. (1976). *The Theory of Partitions*, Addison-Wesley Publishing Company.

[2] Bachelier, L. (1912). *Calcul des Probabilit´es*, Vol. 1. Gauthier-Villars, Paris.

[3] Bruss, F.T., Louchard, G. and Turner, J.W. (2003). *On the N-tower-problem and related problems*, Adv. Appl. Probab. 35, 278-294.

[4] David, G. (2015). *Markov Chain Solution to the 3-Tower Problem*, 3rd International Conference onInformation and Communication Technology-EurAsia (ICT-EURASIA) and 9th International Confer- ence on Research and Practical Issues of Enterprise Information Systems (CONFENIS).

[5] Diaconis, P., Houston-Edwards, K., and Saloff-Coste, L. (2020). *Gambler's ruin estimates on finite inner uniform domains*, Ann. Appl. Probab., 31 (2), 865-895.

[6] Diaconis, P., Ethier, S. (2022). *Gambler's Ruin and the ICM*, Statistical Science, Statist. Sci. 37(3),289-305.

[7] Engel, A. (1993). *The computer solves the three tower problem*, Amer. Math. Monthly 100 (1), 62-64.

[8] Feller, W. (1973). *An introduction to probability theory and its applications*, John Wiley and Sons,third edition.

[9] Ganzfried, S. and Sandholm, T. (2008) *Computing an approximate jam/fold equilibrium for 3- player no-limit Texas hold'em tournaments*, AAMAS08: 7th International Conference on Autonomous Agents and Multi Agent Systems (L. Padgham, D. Parkes, J. M uller, and S. Parsons, eds.) 919-926. International Foundation for Autonomous Agents and Multiagent Systems, Richland, SC.

[10] Grinstead, C. and Snell, J.L. (1997). *Introduction to Probability*, American Mathematical Society, second revised edition.

[11] Kim, M. S. (2005) *Gambler's ruin in many dimensions and optimal strategy in repeated multi-player games with application to poker. Masters thesis, 31, UCLA.*

[12] Malmuth, M. *(1987).* Gambling Theory and Other Topics*, Two Plus Two Publishing, Henderson, NV.*

[13] Marfil, R.I.D. and David, G. *(2020).* On the Placing Probabilities for the Four-Tower Problem UsingRecursions Based on Multigraphs*, Journal of the Mathematical Society of the Philippines, 43:1, 19-32.*

[14] Marfil, R.I.D. and David, G. *(2020).* On the Placing Probabilities and Expected Time Until Ruin for the N -Tower Problem Using Recursions Based on Multigraphs *[Unpublished doctoral dissertation], University of the Philippines Diliman.*

[15] Marfil, R.I.D. and David, G. *(2021).* Solution to the Three-Player Gambler's Ruin with Variable Bet Size Using Recursions Based on Multigraphs*, Science & Engineering Journal, 14:2, 249-256.*

[16] Stirzaker, D. *(1994).* Tower problems and martingales*, The Mathematical Scientist 19, 52-59.*

[17] Stirzaker, D. *(2006).* Three-handed gambler's ruin*, Adv. Appl. Probab. 38 (1), 284–286.*

[18] Swan, Y. and Bruss, F.T. *(2006).* A matrix-analytic approach to the n-player ruin problem*, J. Appl.Probab. 43:3, 755-766.*